

# Structure of Helicity and Global Solutions of Incompressible Navier-Stokes Equation

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## Abstract

In this paper we derive a new energy identity for the three-dimensional incompressible Navier-Stokes equations by a special structure of helicity. The new energy functional is critical with respect to the natural scalings of the Navier-Stokes equations. Moreover, it is conditionally coercive. As an application we construct a family of finite energy smooth solutions to the Navier-Stokes equations whose critical norms can be arbitrarily large.

**Keyword:** Helicity, Navier-Stokes, global solutions, finite energy.

## 1 Introduction

The question of whether a solution of the three-dimensional (3D) incompressible Navier-Stokes equations can develop a finite time singularity from smooth initial data with finite energy is one of the Millennium Prize problems [3]. The only known coercive *a priori* estimate is the Leray-Hopf energy estimate which implies that the 3D Navier-Stokes equations are supercritical with respect to its natural scalings. The latter may be the essence of difficulties of this long standing open problem.

In this paper, by the virtue of a special structure of Helicity, we derive a new *a priori* energy estimate which is critical with respect to the natural scalings for the 3D Navier-Stokes equations. This new energy functional is coercive for a class of initial data. Based on this *a priori* estimate, a family of finite energy global smooth and large solutions can then be constructed. Current known examples of large smooth solutions to the 3D Navier-Stokes equations often assume both the axial symmetricity and the vanishing of swirl component of the velocity, see [5], [7] and [4].

Let us recall that the incompressible Navier-Stokes equations in  $\mathbb{R}_+ \times \mathbb{R}^3$  are:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u, & t > 0, x \in \mathbb{R}^3, \\ \nabla \cdot u = 0, & t > 0, x \in \mathbb{R}^3, \end{cases} \quad (1.1)$$

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where  $u$  is the velocity field of the fluid,  $p$  is the scalar pressure and the constant  $\nu$  is the viscosity. To solve the Navier-Stokes equations (1.1) in  $\mathbb{R}_+ \times \mathbb{R}^3$ , one assumes that the initial data

$$u(0, x) = u_0(x)$$

are divergence-free and possess certain regularity.

The known *a priori* Leray-Hopf energy estimate satisfied by classical solutions of (1.1) is as follows:

$$\sup_{t>0} \|u(t, \cdot)\|_{L^2} \leq \|u_0\|_{L^2}, \quad \nu \int_0^\infty \|\nabla u(t, \cdot)\|_{L^2}^2 dt \leq \frac{1}{2} \|u_0\|_{L^2}^2. \quad (1.2)$$

Recall the natural scalings of the Navier-Stokes equations: if  $(u, p)$  solves (1.1), so does  $(u^\lambda, p^\lambda)$  for any  $\lambda > 0$ , where

$$u^\lambda(t, x) = \lambda u(\lambda t, \lambda x), \quad p^\lambda(t, x) = \lambda^2 p(\lambda t, \lambda x). \quad (1.3)$$

As usual, we assign each  $x_i$  a positive dimension 1,  $t$  a positive dimension 2,  $u$  a negative dimension  $-1$  and  $p$  a negative dimension  $-2$ . A simple dimensional analysis shows that all energy norms in (1.2) have positive dimensions, and thus the Navier-Stokes equations are *supercritical* with respect to the natural scalings. An example of dimensionless norm is  $L_t^\infty(\dot{H}^{\frac{1}{2}})$ , and it will be related to discussions below.

Denote

$$D = \sqrt{-\Delta}.$$

Our starting point is the following new energy identity.

**Theorem 1.1** (Structure of Helicity). *Let  $u \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3))$  solve the incompressible Navier-Stokes equation (1.1). For each  $t \in [0, T)$ , decompose  $u(t, \cdot)$  as in (2.1). Then one has*

$$E_c(u_+) = E_c(u_-) + c_0, \quad \forall t \in [0, T), \quad (1.4)$$

where the constant  $c_0$  is given by

$$c_0 = \frac{1}{2} (\|u_{0+}\|_{\dot{H}^{\frac{1}{2}}}^2 - \|u_{0-}\|_{\dot{H}^{\frac{1}{2}}}^2),$$

and the critical energy  $E_c(u)$  is defined as

$$E_c(u) = \frac{1}{2} \|D^{\frac{1}{2}} u(t, \cdot)\|_{L^2}^2 + \nu \int_0^t \|D^{\frac{1}{2}} \nabla u(s, \cdot)\|_{L^2}^2 ds.$$

The above energy identity which is based on the special structure of helicity gives us an *a priori* estimate. What may be crucial is that this *a priori* estimate (1.4) is critical with respect to the natural scalings (1.3) of the Navier-Stokes equations. Moreover, for the initial data so that either  $u_+$  or  $u_-$  dominates, this *a priori* estimate (1.4) becomes coercive. The proof of Theorem 1.1 will be presented in Section 2.

As an application of Theorem 1.1, we shall construct a family of finite energy smooth large solutions for the 3D incompressible Navier-Stokes equations. Define  $n(\xi)$  as a measurable vector field which is smooth except for finite many singular points and satisfies  $\xi \cdot n(\xi)$ ,  $|n(\xi)| = 1$ . For  $0 < \delta < 1$ , let

$$\alpha \in \mathcal{S}(\mathbb{R}^3), \quad \text{supp } \alpha \subset \{1 - \delta < |\xi| < 1 + \delta\}. \quad (1.5)$$

Assume further that  $\alpha$  vanishes in a neighbourhood of the singular points of  $n(\xi)$  and

$$A = \int_{1-\delta}^{1+\delta} \sup_{\omega \in \mathbb{S}^2} (|\alpha(\lambda\omega)| + |\nabla\alpha(\lambda\omega)|) d\lambda < \infty. \quad (1.6)$$

Define

$$g(x) = \int_{1-\delta < |\xi| < 1+\delta} (n(\xi) \sin(x \cdot \xi) + |\xi|^{-1} \xi \times n(\xi) \cos(x \cdot \xi)) \alpha(\xi) d\xi. \quad (1.7)$$

Let  $\chi \in C_0^\infty(\mathbb{R}^3)$  be a cut-off function such that

$$\chi \equiv 1 \text{ for } |x| \leq 1, \quad \chi \equiv 0 \text{ for } |x| \geq 2, \quad |\nabla^k \chi| \leq 2 \text{ } (0 \leq k \leq 2) \quad (1.8)$$

and

$$\chi_M(x) = \chi\left(\frac{x}{M}\right), \quad M > 0. \quad (1.9)$$

We have the following theorem:

**Theorem 1.2.** *Let  $g$  is given in (1.7) where  $\alpha$  is given in (1.5) and satisfies (1.6). Let  $\chi$  be any standard cut-off function satisfying (1.8)-(1.9). There exist positive constants  $M \geq \delta^{-\frac{1}{2}} \gg 1$  such that the 3D incompressible Navier-Stokes equations (1.1) with the initial data  $u_0 = h_0 + \chi_M g$  are globally well-posed provided that  $\|h_0\|_{H^1} \leq M^{-\frac{1}{2}}$ .*

**Remark 1.3.** An important property of the initial data is that it satisfies  $g_- = 0$  in the above Theorem, which leads one to believe that  $u_+$  would dominate in the evolution of the Navier-Stokes flows. A typical example for  $h_0$  in both Theorem 1.2 and 1.5 can be computed by  $u_0 = \nabla \times (\nabla \times (\chi_M g)) = h_0 + \chi_M g$ . We also note that  $A$  can be arbitrarily large (but finite) in the above Theorem. The latter implies that  $g \in L^\infty$  since  $\widehat{g} \in L^1$  (see (3.1) for details). However,  $\widehat{g}$  may not be an  $L^p$ -function for any  $1 < p \leq 3$  and thus our data may not be small in any critical spaces including  $\dot{B}_{\infty,\infty}^{-1}$ . Besides, the integral interval  $[1-\delta, 1+\delta]$  can be changed to  $\rho[1-\delta, 1+\delta]$  for arbitrary positive  $\rho$  by changing  $d\lambda$  to  $\lambda^2 d\lambda$ , with various appropriate modifications. One may even consider the initial data to be a suitable combination of finitely many  $g$ 's, say,  $g = \sum g_i$ , with each  $\widehat{g}_i$  supported on  $\{\rho_i(1-\delta) \leq |\xi| \leq \rho_i(1+\delta)\}$ . Of course we need impose extra conditions on  $\delta$  and  $\rho_i$ 's. For instance,  $\delta \ll \max_i \{\rho_i/\rho_{i+1}\} \ll 1$  would work.

**Remark 1.4.** There are several other constructions of large, finite energy and smooth solutions for 3D Navier-Stokes equations. Readers may find the following articles to be informative and relevant: Chemin-Gallagher-Paicu [1], Hou-Lei-Li [4] and references therein. We shall emphasize that in these references there may be smallness assumptions imposed on certain dimensionless norms of unknowns or a part of unknowns. For instance, in [1], the  $L^2$  norm of  $|D_z^{\frac{1}{2}} u_1| + |D_z^{\frac{1}{2}} u_2| + |D_{xy}^{-1} D_z^{3/2} u_3|$  is small. Here  $\widehat{D_z f}(\xi) = |\xi_3| \widehat{f}(\xi)$  and  $\widehat{D_{xy}^{-1} f}(\xi) = (|\xi_1| + |\xi_2|)^{-1} \widehat{f}(\xi)$ .

A simple and more typical example of  $g$  in the initial data is

$$g_0(x) = \int_{\mathbb{S}^2} (n(\xi) \sin(x \cdot \xi) + |\xi|^{-1} \xi \times n(\xi) \cos(x \cdot \xi)) \beta(\xi) d\sigma_\xi. \quad (1.10)$$

Here  $\beta \in C^1(\mathcal{S}^2)$  is a given function which vanishes in a neighbourhood of the singular point of  $n(\xi)$ . This turns out to be the steady state Beltrami flow, which has already been observed in the book of Majda and Bertozzi in [6] (see also [2]):

$$\nabla \cdot g_0 = 0, \quad \nabla \times g_0 = g_0.$$

We refer to section 3 below for a more detailed discussion. Here we can formulate the following theorem:

**Theorem 1.5.** *Let  $\beta \in C^1(\mathcal{S}^2)$  and  $\chi$  be any standard cut-off function satisfying (1.8)-(1.9). There exists a large positive constant  $M$  such that the 3D incompressible Navier-Stokes equations (1.1) with the initial data  $u_0 = h_0 + \chi_M g_0$  are globally well-posed provided that  $M \geq M_0$  and  $\|h_0\|_{H^1} \leq M^{-\frac{1}{2}}$ .*

The proof of Theorem 1.5 is exactly the same as that for Theorem 1.2. Our proof of Theorem 1.2 is elementary and it is based on a perturbation argument along with a standard cut-off technique. A key point is a decay estimate in the spatial directions of such family of initial data. Let us briefly explain the main idea involved. We let  $v$  be obtained from the heat flow with initial data  $g$  or  $g_0$ . Write the solution as  $u = h + v\chi_M$ . Then we try to solve for  $h$ . Note that  $h$  is not divergence-free (see equation (4.4)). Then main difficulty in solving for  $h$  is that the "force term" may not be small. Indeed, it is easy to check that one of the forcing terms in  $h$ -equation (see (4.4) and (4.6)) is  $\nabla(\frac{1}{2}\chi_M^2|v|^2)$ , which is not small in  $L_t^2(\dot{H}^{-\frac{1}{2}})$  (or  $L_t^2(L^3)$  norm of  $\frac{1}{2}\chi_M^2|v|^2$ ). Thus the standard parabolic estimate doesn't give an  $L_t^2(\dot{H}^{\frac{3}{2}}) \cap L_t^\infty(\dot{H}^{\frac{1}{2}})$  estimate of the perturbation  $h$  from  $v\chi_M$ . In addition,  $\|\nabla \cdot h\|_{L_t^\infty(\dot{H}^{-\frac{1}{2}})}$  (and thus  $\|h\|_{L_t^\infty(\dot{H}^{\frac{1}{2}})}$ ) may not be small. Details of these will be discussed in section 4.

The remaining part of the paper is organized as follows. We will first prove Theorem 1.1 by the virtue of the structure of the helicity in section 2. In section 3 we study the decay properties of the initial data given in (1.7) and (1.10). Then we prove Theorem 1.2 and Theorem 1.5 in section 4.

## 2 Structure of Helicity

It is well-known that the helicity  $\int u \cdot \omega dx$  is conserved in time for 3D incompressible Euler equations. However, due to the presence of dissipation, the helicity is not conserved for 3D incompressible Navier-Stokes equations. It is not clear how to make use of such a quantity without positivity of its integrand even in the case of Euler equations.

In this section we will explore a structure of the helicity which is inherent by smooth solutions to the 3D incompressible Euler or Navier-Stokes equations. Our proof of Theorem 1.1 is based on a strongly orthogonal decomposition of the velocity vector which is stated in Proposition 2.2. Similar conclusions were studied earlier by P. Constantin and A. Majda for incompressible Euler equations in [2].

Let  $u$  be a divergence-free vector field. We make the following decomposition:

$$u = u_+ + u_-, \tag{2.1}$$

where

$$u_+ = \frac{1}{2}(u + D^{-1}\nabla \times u)$$

and

$$u_- = \frac{1}{2}(u - D^{-1}\nabla \times u).$$

We have the following proposition:

**Proposition 2.1.** *Let  $u \in H^1(\mathbb{R}^3)$  be a 3D divergence-free vector field and be decomposed into  $u_+$  and  $u_-$  as in (2.1). Then the following identities hold:*

$$\nabla \times u_+ = Du_+, \quad \nabla \times u_- = -Du_-.$$

*Proof.* We only need to show the first identity. The second one is similar. The proof is straightforward. In fact, due to the divergence-free property of  $u$ , it is clear that

$$\begin{aligned} \nabla \times u_+ &= \frac{1}{2}(\nabla \times u + D^{-1}\nabla \times \nabla \times u) \\ &= \frac{1}{2}(\nabla \times u - D^{-1}\Delta u) \\ &= \frac{1}{2}D(D^{-1}\nabla \times u + u) \\ &= Du_+. \end{aligned}$$

□

The following proposition shows that  $u_+$  and  $u_-$  are strongly orthogonal to each other.

**Proposition 2.2.** *Let  $m, k \geq 0$  be any integers and  $u \in C^m([0, T], H^k(\mathbb{R}^3))$ . Suppose that for each  $t \in [0, T]$ ,  $u(t, \cdot)$  is divergence-free. Decompose  $u(t, \cdot)$  into  $u_+(t, \cdot)$  and  $u_-(t, \cdot)$  as in (2.1). Then for all integers  $m_1, m_2$  and  $k_1, k_2$  with  $m_1 + m_2 \leq m$  and  $k_1 + k_2 \leq k$ , we have*

$$\int D^{m_1} \partial_t^{k_1} u_+ \cdot D^{m_2} \partial_t^{k_2} u_- dx \equiv 0.$$

*Proof.* Without loss of generality, we may assume that  $m_2 < m$ . By Proposition 2.1, one has

$$u_+ = D^{-1}\nabla \times u_+.$$

Consequently, one has

$$\begin{aligned} &\int D^{m_1} \partial_t^{k_1} u_+ \cdot D^{m_2} \partial_t^{k_2} u_- dx \\ &= \int D^{m_1} \partial_t^{k_1} D^{-1}\nabla \times u_+ \cdot D^{m_2} \partial_t^{k_2} u_- dx \\ &= \int D^{m_1} \partial_t^{k_1} D^{-1} u_+ \cdot D^{m_2} \partial_t^{k_2} \nabla \times u_- dx \\ &= - \int D^{m_1} \partial_t^{k_1} D^{-1} u_+ \cdot D^{m_2+1} \partial_t^{k_2} u_- dx \\ &= - \int D^{m_1} \partial_t^{k_1} u_+ \cdot D^{m_2} \partial_t^{k_2} u_- dx, \end{aligned}$$

which implies the result in the proposition. □

We are ready now to prove our structural theorem, i.e. Theorem 1.1, for the helicity of solutions to the incompressible Navier-Stokes equations.

*Proof.* We first notice that by integration by parts and the divergence-free property of  $u$ , there holds

$$\begin{aligned}\frac{d}{dt} \int u \cdot \omega dx &= \int u_t \cdot \omega dx + \int (\nabla \times u_t) \cdot u dx \\ &= 2 \int u_t \cdot \omega dx, \quad \text{for } \omega = \nabla \times u.\end{aligned}$$

Consequently, there holds the following identity for helicity of the incompressible Navier-Stokes equations (1.1):

$$\frac{d}{dt} \int u \cdot \omega dx = 2\nu \int \Delta u \cdot \omega dx. \quad (2.2)$$

Applying the decomposition in (2.1) and using Proposition 2.1, we obtain that

$$\begin{aligned}\int u \cdot \omega dx &= \frac{1}{4} \int (u_+ + u_-) \cdot (\nabla \times u_+ + \nabla \times u_-) dx \\ &= \frac{1}{4} \int (u_+ + u_-) \cdot (Du_+ - Du_-) dx.\end{aligned}$$

By Proposition 2.2, we further deduce that

$$\begin{aligned}\int u \cdot \omega dx &= \frac{1}{4} \int (u_+ \cdot Du_+ - u_- \cdot Du_-) dx \\ &= \frac{1}{4} (\|D^{\frac{1}{2}} u_+\|_{L^2}^2 - \|D^{\frac{1}{2}} u_-\|_{L^2}^2).\end{aligned} \quad (2.3)$$

Similarly, we have

$$\int \Delta u \cdot \omega dx = -\frac{1}{4} (\|D^{\frac{3}{2}} u_+\|_{L^2}^2 - \|D^{\frac{3}{2}} u_-\|_{L^2}^2). \quad (2.4)$$

Plugging (2.3) and (2.4) into (2.2), we arrive at

$$\begin{aligned}&\frac{d}{dt} \left( \frac{1}{2} \|D^{\frac{1}{2}} u_+(t)\|_{L^2}^2 + \nu \int_0^t \|D^{\frac{3}{2}} u_+(s)\|_{L^2}^2 ds \right) \\ &= \frac{d}{dt} \left( \frac{1}{2} \|D^{\frac{1}{2}} u_-(t)\|_{L^2}^2 + \nu \int_0^t \|D^{\frac{3}{2}} u_-(s)\|_{L^2}^2 ds \right).\end{aligned}$$

Integrating the above differential inequality with respect to time, one can complete the proof of the theorem.  $\square$

### 3 Decay Properties of Data

First of all, let us rewrite the data in (1.7) as

$$\begin{aligned}
g(x) &= \frac{1}{2} \int_{1-\delta < |\xi| < 1+\delta} (-in(\xi) + |\xi|^{-1} \xi \times n(\xi)) \alpha(\xi) e^{ix \cdot \xi} d\xi \\
&\quad + \frac{1}{2} \int_{1-\delta < |\xi| < 1+\delta} (in(\xi) + |\xi|^{-1} \xi \times n(\xi)) \alpha(\xi) e^{-ix \cdot \xi} d\xi \\
&= \frac{1}{2} \mathcal{F}^{-1} [(-in(\xi) + |\xi|^{-1} \xi \times n(\xi)) \alpha(\xi)](x) \\
&\quad + \frac{1}{2} \mathcal{F}^{-1} [(in(\xi) + |\xi|^{-1} \xi \times n(\xi)) \alpha(\xi)](-x).
\end{aligned} \tag{3.1}$$

It is easy to check that

$$\nabla \cdot g = 0, \quad \nabla \times g = Dg. \tag{3.2}$$

Hence, one has

$$g_- = 0, \quad g = g_+. \tag{3.3}$$

Moreover, there holds

$$\|g\|_{L^\infty} + \|\nabla g\|_{L^\infty} \lesssim \|\widehat{g}\|_{L^1} + \|\widehat{\nabla g}\|_{L^1} \lesssim A. \tag{3.4}$$

Next, we study the spatial decay properties of  $g$  given in (1.7). For each  $x$  with  $|x| \neq 0$ , let  $B(x)$  be an orthogonal matrix such that

$$x = B\bar{x}, \quad \bar{x} = \begin{pmatrix} 0 \\ 0 \\ |x| \end{pmatrix}.$$

We use the sphere coordinate to parameterize  $y$  as follows:

$$B^T y = \tilde{n}(y) = \begin{pmatrix} \sin \phi \cos \theta \\ \sin \phi \sin \theta \\ \cos \phi \end{pmatrix}, \quad 0 \leq \phi \leq \pi, -\pi \leq \theta \leq \pi.$$

We compute that

$$\begin{aligned}
g(x) &= \int \lambda^2 d\lambda \int_{\mathbb{S}^2} [n(y) \sin \langle \tilde{n}(y), \bar{x} \rangle \\
&\quad + y \times n(y) \cos \langle \tilde{n}(y), \bar{x} \rangle] a(\lambda y) d\sigma_y \\
&= \int_0^\pi \int_0^{2\pi} [n(y) \sin(\kappa|x| \cos \phi) + y \times n(y) \cos(|x| \cos \phi)] \sin \phi d\theta a(\lambda y) d\phi \\
&= \frac{1}{|x|} \int_0^{2\pi} d\theta \int_0^\pi a(\lambda y) [n(y) d \cos(|x| \cos \phi) - y \times n(y) d \sin(|x| \cos \phi)] \\
&= \frac{1}{|x|} \int_0^{2\pi} a(\lambda y) [n(y) \cos(|x| \cos \phi) - y \times n(y) \sin(|x| \cos \phi)] \Big|_{\phi=0}^{\phi=\pi} d\theta \\
&\quad - \frac{1}{|x|} \int_0^{2\pi} d\theta \int_0^\pi [\cos(|x| \cos \phi) d(n(y) a(\lambda y)) \\
&\quad - \sin(|x| \cos \phi) d(a(\lambda y) y \times n(y))].
\end{aligned}$$

Since, by (3.4),  $|\partial_\phi[a(\lambda y)y \times n(y)]| + |\partial_\phi[n(y)a(\lambda y)]| \lesssim A$ , one has

$$|g(x)| \leq \frac{A}{|x|}.$$

A similar bound can also be verified for  $\nabla g$ . Hence, we have

$$|g(x)| + |\nabla g(x)| \lesssim \frac{A}{1 + |x|}. \quad (3.5)$$

A similar calculation as above shows that  $g_0$  given in (1.10) satisfies

$$\nabla \cdot g_0 = 0, \quad \nabla \times g_0 = g_0, \quad |g_0(x)| + |\nabla g_0(x)| \lesssim \frac{1}{1 + |x|}.$$

## 4 Constructing Solutions by Cut-off and Perturbation

In this section we construct the global smooth solutions to the 3D Navier-Stokes equations with finite energy using the standard cut-off and perturbation arguments.

First of all, let  $v(t, x)$  be the solution of the heat equation

$$v_t = \nu \Delta v, \quad v(0, x) = v_0. \quad (4.1)$$

If  $v_0 = g$  which is given in (1.7), then by (3.2) (it is preserved by the heat flow in (4.1)), one has

$$\begin{aligned} v_t + v \cdot \nabla v + \nabla \left( -\frac{1}{2} |v|^2 \right) - \nu \Delta v \\ = -v \times (\nabla \times v) = -v \times Dv \\ = -v \times (D - 1)v. \end{aligned} \quad (4.2)$$

In this case,  $v$  is a solution of the Navier-Stokes equations with a forcing term  $-v \times (D - 1)v$ . Clearly, one has the following estimate:

$$|v(x)| + |\nabla v(x)| \lesssim \frac{Ae^{-\nu t/2}}{1 + |x|}. \quad (4.3)$$

Indeed, choosing  $\beta \in C_0^\infty$  so that  $\beta = 1$  on the support of  $\alpha$  and  $\beta = 0$  if  $|\xi| \geq 1 + 2\delta$  and  $|\xi| \leq 1 - 2\delta$ , one has  $v(t, x) = e^{-\nu t/2} \mathcal{F}^{-1} \left( e^{-\nu \sqrt{|\xi|^2 - 1/2} t} \beta(\xi) \right) * g(x)$ . Note that  $1/2 \leq |\xi|^2 - 1/2 \leq (1 + 2\delta)^2 - 1/2$  in the kernel  $\mathcal{F}^{-1} \left( e^{-\nu \sqrt{|\xi|^2 - 1/2} t} \beta(\xi) \right)$ . Then one can easily verify (4.3) by using (3.5).

If  $v_0 = g_0$  which is given in (1.10), then the forcing term in (4.2) vanishes and the estimate (4.3) still holds. So the proof of Theorem 1.5 can be carried out in the same way as that of Theorem 1.2. Below we will only present the proof for Theorem 1.2.

Suppose that  $u$  is the unique local smooth solution of the Navier-Stokes equations with initial data  $u(0, x) = h_0 + \chi_M g(x)$ . Here  $\|h_0\|_{H^1} \leq M^{-\frac{1}{2}}$ . The associated pressure is  $p = -\Delta^{-1} \nabla \cdot [\nabla \cdot (u \otimes u)]$ . To show that  $u(t, x)$  is a global smooth solution, it is sufficient to prove an *a priori* estimate for  $\|u(t, \cdot)\|_{H^1}$  for all  $t > 0$ . Define

$$h = u - \chi_M v.$$



It is easy to see that  $h$  is governed by

$$\begin{cases} h_t + h \cdot \nabla h + \nabla p = \Delta h + f, \\ \nabla \cdot h = -v \cdot \nabla \chi_M, \quad h(0, x) = h_0(x), \end{cases} \quad (4.4)$$

where  $f$  is given by

$$\begin{aligned} f &= -\chi_M(v_t - \Delta v) - h \cdot \nabla(\chi_M v) - \chi_M v \cdot \nabla h \\ &\quad + v \Delta \chi_M + 2(\nabla \chi_M \cdot \nabla)v - \chi_M(v \cdot \nabla \chi_M)v - \chi_M^2 v \cdot \nabla v \\ &= v \Delta \chi_M + 2(\nabla \chi_M \cdot \nabla)v - \chi_M(v \cdot \nabla \chi_M)v - \chi_M^2 v \cdot \nabla v \\ &\quad - h \cdot \nabla(\chi_M v) - \chi_M v \cdot \nabla h. \end{aligned}$$

Here and in what follows we will set  $\nu$  to be 1.

Taking the  $L^2$  inner product of (4.4) with  $h$ , we have

$$\frac{1}{2} \frac{d}{dt} \|h\|_{L^2}^2 + \|\nabla h\|_{L^2}^2 = \frac{1}{2} \int |h|^2 \nabla \cdot h dx + \int (p \nabla \cdot h + f h) dx.$$

Now let us use the expressions for  $p$  and  $f$  to rewrite that

$$\begin{aligned} &\int (p \nabla \cdot h + f h) dx \\ &= \int \left( -\Delta^{-1} \nabla \cdot [h \cdot \nabla h - v \Delta \chi_M - 2(\nabla \chi_M \cdot \nabla)v + \chi_M v \cdot \nabla h + h \cdot \nabla(\chi_M v)] \right. \\ &\quad \left. + \chi_M^2 v \cdot \nabla v + (\chi_M v) v \cdot \nabla \chi_M \right) \nabla \cdot h + [v \Delta \chi_M + 2(\nabla \chi_M \cdot \nabla)v \\ &\quad - \chi_M(v \cdot \nabla \chi_M)v - \chi_M^2 v \cdot \nabla v - h \cdot \nabla(\chi_M v) - \chi_M v \cdot \nabla h] h \Big) dx \\ &= \int (h \cdot \nabla h) \Delta^{-1} \nabla \nabla \cdot h dx + \int (\chi_M^2 v \cdot \nabla v) (\Delta^{-1} \nabla \nabla \cdot h - h) dx \\ &\quad + \int \left( [\chi_M v \cdot \nabla h + h \cdot \nabla(\chi_M v) + (\chi_M v) v \cdot \nabla \chi_M] (\Delta^{-1} \nabla \nabla \cdot h - h) \right. \\ &\quad \left. + [-v \Delta \chi_M + 2 \nabla_j (\nabla_j \chi_M v)] (\Delta^{-1} \nabla \nabla \cdot h - h) \right) dx. \end{aligned}$$

Consequently, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|h\|_{L^2}^2 + \|\nabla h\|_{L^2}^2 &= -\frac{1}{2} \int |h|^2 v \cdot \nabla \chi_M dx \\ &\quad + \int (h \cdot \nabla h) \Delta^{-1} \nabla \nabla \cdot h dx + \int (\chi_M^2 v \cdot \nabla v) (\Delta^{-1} \nabla \nabla \cdot h - h) dx \\ &\quad + \int \left( [\chi_M v \cdot \nabla h + h \cdot \nabla(\chi_M v) + (\chi_M v) v \cdot \nabla \chi_M] \Delta^{-1} \nabla \times \nabla \times h \right. \\ &\quad \left. + [-v \Delta \chi_M + 2 \nabla_j (\nabla_j \chi_M v)] \Delta^{-1} \nabla \times \nabla \times h \right) dx. \end{aligned} \quad (4.5)$$

We need estimate the right hand side of (4.5) term by term. First of all, by (4.3), by Sobolev imbedding inequality, it is easy to see that

$$\left| \frac{1}{2} \int |h|^2 v \cdot \nabla \chi_M dx \right| \lesssim M^{-1} \|v\|_{L^\infty} \|h\|_{L^2}^2 \lesssim M^{-1} e^{-t/2} \|h\|_{L^2}^2.$$

Next, for the first term of the second line on the right hand side of (4.5), one can simply estimate that

$$\left| \int (h \cdot \nabla h) \Delta^{-1} \nabla \nabla \cdot h dx \right| \lesssim \|h\|_{L^6} \|\nabla h\|_{L^2} \|\Delta^{-1} \nabla \nabla \cdot h\|_{L^3} \lesssim \|h\|_{L^3} \|\nabla h\|_{L^2}^2.$$

Here we used the Sobolev imbedding  $\|g\|_{L^6} \lesssim \|\nabla g\|_{L^2}$  and the standard Calderon-Zygmund theory  $\|Zg\|_{L^p} \lesssim \|g\|_{L^p}$  for Riesz operator  $Z$  and  $1 < p < \infty$ . To treat the second term of the second line on the right hand side of (4.5), we first write that

$$v \cdot \nabla v = -v \times (\nabla \times v) + \frac{1}{2} \nabla |v|^2.$$

Using (4.2), we have

$$\chi_M^2 v \cdot \nabla v = -\chi_M^2 v \times (D-1)v - \chi_M |v|^2 \nabla \chi_M + \nabla \left( \frac{1}{2} \chi_M^2 |v|^2 \right). \quad (4.6)$$

Consequently, one has

$$\begin{aligned} & \left| \int (\chi_M^2 v \cdot \nabla v) (\Delta^{-1} \nabla \nabla \cdot h - h) dx \right| \\ &= \left| \int [\chi_M^2 v \times (D-1)v + \chi_M |v|^2 \nabla \chi_M] \Delta^{-1} \nabla \times \nabla \times h dx \right| \\ &\lesssim (M^{-1} \|v\|_{L^{12/5}(|x| \leq M)}^2 + \|(D-1)v\|_{L^\infty} \|v\|_{L^{6/5}(|x| \leq M)}) \|\Delta^{-1} \nabla \times \nabla \times h\|_{L^6} \\ &\lesssim (M^{-1/2} + \|(|\xi| - 1)\widehat{v}\|_{L^1} M^{3/2}) e^{-t} \|\Delta^{-1} \nabla \times \nabla \times h\|_{L^6} \\ &\lesssim (M^{-1} + \delta^2 M^3) e^{-2t} + \frac{1}{16} \|\nabla h\|_{L^2}^2. \end{aligned}$$

Now let us estimate the third line on the right hand side of (4.5). As above, we have

$$\begin{aligned} & \left| \int (\chi_M v \cdot \nabla h + h \cdot \nabla (\chi_M v)) \Delta^{-1} \nabla \times \nabla \times h dx \right| \\ &\lesssim (\|\chi_M v\|_{L^\infty} \|\nabla h\|_{L^2} + \|\nabla (\chi_M v)\|_{L^\infty} \|h\|_{L^2}) \|\Delta^{-1} \nabla \times \nabla \times h\|_{L^2} \\ &\lesssim e^{-t/2} \|h\|_{L^2}^2 + \frac{1}{16} \|\nabla h\|_{L^2}^2, \end{aligned}$$

and

$$\begin{aligned} & \left| \int (\chi_M v) v \cdot \nabla \chi_M \Delta^{-1} \nabla \times \nabla \times h dx \right| \\ &\lesssim M^{-1} \|\sqrt{\chi_M} v\|_{L^{\frac{12}{5}}}^2 \|\Delta^{-1} \nabla \times \nabla \times h\|_{L^6} \\ &\lesssim M^{-1} e^{-2t} + \frac{1}{16} \|\nabla h\|_{L^2}^2. \end{aligned}$$

For the last line, we have

$$\left| \int v \Delta \chi_M h dx \right| \lesssim M^{-2} \|v\|_{L^{\frac{6}{5}}(|x| \leq 2M)} \|h\|_{L^6} \lesssim M^{-1} e^{-t} + \frac{1}{16} \|\nabla h\|_{L^2}^2.$$

Moreover, using integration by parts, we have

$$\left| \int 2\nabla_j(\nabla_j \chi_M v) h dx \right| \lesssim M^{-1} \|v\|_{L^2(|x| \leq 2M)} \|\nabla h\|_{L^2} \lesssim M^{-1} e^{-t} + \frac{1}{16} \|\nabla h\|_{L^2}^2.$$

Inserting all the above estimates into (4.5), we arrive at

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|h\|_{L^2}^2 + \left( \frac{5}{16} - C \|h\|_{L^3} \right) \|\nabla h\|_{L^2}^2 \\ & \lesssim e^{-t/2} \|h\|_{L^2}^2 + (M^{-1} + \delta^2 M^3) e^{-t}. \end{aligned} \quad (4.7)$$

Now let us apply the curl operator to (4.4) and then take the  $L^2$  inner product of the resulting equation with curl  $h$  to get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla \times h\|_{L^2}^2 + \|\nabla \nabla \times h\|_{L^2}^2 \\ & = - \int \nabla \times (h \cdot \nabla h) \nabla \times h dx + \int (\nabla \times f) \cdot (\nabla \times h) dx. \end{aligned} \quad (4.8)$$

We first deal with the first term on the right hand side of (4.8). Using integration by parts and Hodge decomposition, we estimate that

$$\begin{aligned} \left| \int \nabla \times (h \cdot \nabla h) \nabla \times h dx \right| & \lesssim \|h\|_{L^3} \|\nabla h\|_{L^6} \|\nabla \times \nabla \times h\|_{L^2} \\ & \lesssim \|h\|_{L^3} \|\nabla \nabla \times h\|_{L^2}^2 + \|h\|_{L^3} \|\nabla \cdot h\|_{L^6} \|\nabla \times \nabla \times h\|_{L^2}. \end{aligned}$$

Recall the second equation in (4.4), one has

$$\|\nabla \cdot h\|_{L^6} = \|v \cdot \nabla \chi_M\|_{L^6} \lesssim M^{-1} e^{-t/2}.$$

Using interpolation  $\|h\|_{L^3} \lesssim \|h\|_{L^2}^{\frac{1}{2}} \|\nabla h\|_{L^2}^{\frac{1}{2}}$ , one finally has

$$\begin{aligned} & \left| \int \nabla \times (h \cdot \nabla h) \nabla \times h dx \right| \\ & \lesssim \|h\|_{L^3} \|\nabla \nabla \times h\|_{L^2}^2 + \frac{1}{16} (\|\nabla h\|_{L^2}^2 + \|\nabla \nabla \times h\|_{L^2}^2) + M^{-4} e^{-2t} \|h\|_{L^2}^2. \end{aligned}$$

For the second term on the right hand side of (4.8), we first write it as follows:

$$\begin{aligned} & \int (\nabla \times f) \cdot (\nabla \times h) dx \\ & = \int \nabla \times (v \Delta \chi_M + 2\nabla \chi_M \cdot \nabla v - \chi_M (v \cdot \nabla \chi_M) v) \nabla \times h dx \\ & \quad + \int \nabla \times (\chi_M^2 v \times (D-1)v - \frac{1}{2} \chi_M^2 \nabla |v|^2) \nabla \times h dx \\ & \quad - \int \nabla \times (h \cdot \nabla (\chi_M v) + \chi_M v \cdot \nabla h) \nabla \times h dx. \end{aligned} \quad (4.9)$$

The first line on the right hand side of (4.9) is treated as follows:

$$\begin{aligned}
& \left| \int \nabla \times (v \Delta \chi_M + 2 \nabla \chi_M \nabla v - \chi_M (v \cdot \nabla \chi_M) v) \nabla \times h dx \right| \\
& \lesssim (M^{-2} \|v\|_{L^2(|x| \leq 2M)} + M^{-1} \|\nabla v\|_{L^2(|x| \leq 2M)} + M^{-1} \|v\|_{L^4}^2) \|\nabla \nabla \times h\|_{L^2} \\
& \leq M^{-1} e^{-t} + \frac{1}{16} \|\nabla \nabla \times h\|_{L^2}^2.
\end{aligned}$$

For the second term on the right hand side of (4.9), we first have

$$\begin{aligned}
\left| \int \nabla \times \left( \frac{1}{2} \chi_M^2 \nabla |v|^2 \right) \nabla \times h dx \right| &= \left| \int \nabla \times \left( \frac{1}{2} |v|^2 \nabla \chi_M^2 \right) \nabla \times h dx \right| \\
&\lesssim M^{-2} e^{-2t} + \frac{1}{16} \|\nabla \nabla \times h\|_{L^2}^2.
\end{aligned}$$

On the other hand, we estimate that

$$\begin{aligned}
& \left| \int \nabla \times (\chi_M^2 v \times (D-1)v) \nabla \times h dx \right| \\
& \lesssim \|v\|_{L^2(|x| \leq M)} \|(D-1)v\|_{L^\infty} \|\nabla \nabla \times h\|_{L^2} \\
& \lesssim \delta^2 M e^{-2t} + \frac{1}{16} \|\nabla \nabla \times h\|_{L^2}^2
\end{aligned}$$

We estimate the last line on the right hand side of (4.9) as follows:

$$\begin{aligned}
& \left| \int \nabla \times (-h \cdot \nabla (\chi_M v) - \chi_M v \cdot \nabla h) \nabla \times h dx \right| \\
& \leq (\|h\|_{L^2} \|\nabla (\chi_M v)\|_{L^\infty} + \|\chi_M v\|_{L^\infty} \|\nabla h\|_{L^2}) \|\nabla \nabla \times h\|_{L^2} \\
& \lesssim e^{-t/2} (\|h\|_{L^2} + \|\nabla h\|_{L^2}) \|\nabla \nabla \times h\|_{L^2} \\
& \lesssim e^{-t} (\|h\|_{L^2}^2 + \|\nabla \times h\|_{L^2}^2 + \|v \cdot \nabla \chi_M\|_{L^2}^2) + \frac{1}{16} \|\nabla \nabla \times h\|_{L^2}^2 \\
& \lesssim e^{-t} (\|h\|_{L^2}^2 + \|\nabla \times h\|_{L^2}^2) + M^{-1} e^{-2t} + \frac{1}{16} \|\nabla \nabla \times h\|_{L^2}^2.
\end{aligned}$$

We finally arrive at

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\nabla \times h\|_{L^2}^2 + \left( \frac{5}{16} - C \|h\|_{L^3} \right) \|\nabla \nabla \times h\|_{L^2}^2 \\
& \lesssim e^{-t} (\|h\|_{L^2}^2 + \|\nabla \times h\|_{L^2}^2) + \frac{1}{16} \|\nabla h\|_{L^2}^2 \\
& \quad + (\delta^2 M + M^{-1}) e^{-t}.
\end{aligned} \tag{4.10}$$

Now let us add up (4.7) and (4.10) to yield that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (\|h\|_{L^2}^2 + \|\nabla \times h\|_{L^2}^2) + \left( \frac{3}{8} - C \|h\|_{L^3} \right) (\|\nabla h\|_{L^2}^2 + \|\nabla \nabla \times h\|_{L^2}^2) \\
& \lesssim e^{-t/2} (\|h\|_{L^2}^2 + \|\nabla \times h\|_{L^2}^2) + (\delta^2 M + \delta^2 M^3 + M^{-1}) e^{-t}.
\end{aligned} \tag{4.11}$$

If there holds

$$\delta \lesssim M^{-2}, \quad C \|h\|_{L^3} \leq \frac{3}{8}, \tag{4.12}$$

on some time interval  $0 \leq t \leq T$ , then (4.11) implies

$$(\|h(t, \cdot)\|_{L^2}^2 + \|\nabla \times h(t, \cdot)\|_{L^2}^2) + \int_0^T (\|\nabla h\|_{L^2}^2 + \|\nabla \nabla \times h\|_{L^2}^2) ds \lesssim M^{-1} \quad (4.13)$$

on the same time interval. Since

$$C\|h\|_{L^3} \lesssim (\|h\|_{L^2} + \|\nabla \times h\|_{L^2} + \|\nabla \cdot h\|_{L^2}) \lesssim M^{-\frac{1}{2}}, \quad 0 \leq t \leq T, \quad (4.14)$$

one sees that the second inequality in (4.12) is verified provided that  $M$  is sufficiently large. A standard continuation argument simply implies that (4.13) holds for all time  $t \geq 0$ . Then one has

$$\|h\|_{H^1} \lesssim (\|h\|_{L^2} + \|\nabla \times h\|_{L^2} + \|\nabla \cdot h\|_{L^2}) \lesssim M^{-\frac{1}{2}}, \quad (4.15)$$

for all time  $t \geq 0$ . Since  $u = h + \chi_M v$ , one has  $u \in L^\infty(0, T; H^1)$ , which is sufficient for the global regularity of  $u$ .

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